Online Logistic Regression

For \( t = 1, \ldots, T \):
- receive example \( x_t \in \mathbb{R}^d \) (arg \( \{x_t\}_t \subseteq [1, K] \))
- predict a "score" \( z_t \in \mathbb{R} \)
- observe true label \( y_t \in \{1, -1\} \)
- suffer log loss \(\ell(z_t, y_t) = -\log(1 + e^{-y_t z_t})\)

Goal: minimize regret

\[
\text{Regret} = \sum_{t=1}^{T} \ell(z_t, y_t) - \min_{z} \sum_{t=1}^{T} \ell(z, y_t)
\]

Optimal regret in terms of \( d, T, K \)?

Algorithmic Approach

### Mixability of logistic loss

Vovk's \(\ell \in [1, K] \) is mixable if:
- \(\forall \delta > 0\) there exists a constant \(\eta\) such that:
- \(\text{Regret} = \sum_{t=1}^{T} \ell(z_t, y_t) \leq \eta \cdot T\)

Vovk's Aggregating Algorithm

\[
\frac{1}{T} \sum_{t=1}^{T} \ell(z_t, y_t) \leq \frac{1}{T} \sum_{t=1}^{T} \ell(z_{t-1}^*, y_t) + \frac{1}{T} \sum_{t=1}^{T} \ell(z_t, y_t)
\]

### Application 1: Bandit Multiclass

For \(t = 1, \ldots, T\):
- receive example \( x_t \in \mathbb{R}^d \)
- predict a label \( y_t \in [K] \)
- observe true label \( y_t \in [K] \)

Goal: minimize \(\text{Regret}\) of \(\hat{y}_t\) in \([K] \)

Weak Learning Condition (JGT'17)

Let a weak learner \( h : [K] \rightarrow \{0, 1\} \) be such that for all \( w^* \in [K] \):
- \(\sum_{y \neq w^*} \Pr[\hat{y} = y | x, w^*] \leq \frac{1}{2} + \epsilon\)

Prior work: BKL'15: gave optimal algorithm and separate adaptive algorithm with sub-optimal sample complexity.

Our result:
- Adaptive online boosting with optimal sample complexity.

Extension 2: General Function Classes

### General function class setup:

General instance space \(X\), benchmark class \( F \subseteq (X \rightarrow \mathbb{R})\).

\[
\text{Regret} = \sum_{t=1}^{T} \ell(h_t, y_t) - \min_{h \in F} \sum_{t=1}^{T} \ell(h, y_t)
\]

Minimax regret: \(\mathbb{V}_n(F)\).

### Definition: Tree

A set \( V \) of \(\mathbb{R}^2\)-valued \(K\)-ary trees of depth \( n \) is a sequence \( x = (x_1, \ldots, x_n) \) of mappings with \( x_i \in [K]^{t-1} \rightarrow \mathbb{R}^2 \).

### Extension 1: Statistical Learning

### Setup:

- Receive samples \((x_1, y_1), \ldots, (x_n, y_n)\) i.i.d. from unknown distribution \(D\).

Goal: Output predictor \( \hat{g} : X \rightarrow \mathbb{R} \) such that with prob \(\geq 1 - \delta\):

\[
\mathbb{E}_{D} \ell(\hat{g}(x), y) - \inf_{h \in F} \mathbb{E}_{D} \ell(h(x), y) \leq \text{ExcessRisk}(d, K, \delta, D)
\]

Improper learning: \( \hat{g} \) is possibly non-linear.

**Theorem:** For every \( \delta \), there exists predictor \( \hat{g} \) such that with probability at least \( 1 - \delta \):

\[
\mathbb{E}_{D} \ell(\hat{g}(x), y) - \inf_{h \in F} \mathbb{E}_{D} \ell(h(x), y) \leq O\left(\frac{\text{ExcessRisk}(d, K, \delta, D)}{\delta}\right)
\]

Approach: Adapt the "boosting the confidence" strategy from Mehta'17.